MATH5011 Exercise 5

Problems 2 and 3 are optional. In Problems 6 and 7 we continue the study of the n-dimensional Lebsegue measure starting in Exercise 3.

(1) Let $f : X \to (-\infty, \infty]$ be l.s.c. (lower semi-continuous) where X is a topological space. Show that

(a) αf is l.s.c. $\forall \alpha \geq 0$,

(b) g l.s.c $\Rightarrow \min\{f, g\}$ l.s.c,

(c) f_α l.s.c $\Rightarrow \sup\limits_\alpha f_\alpha$ l.s.c.,

(d) g l.s.c. $\Rightarrow f + g$ l.s.c.

(e) $\infty > f > 0 \Rightarrow 1/f$ is u.s.c..

(2) Let X be a locally compact Hausdorff space. Let $f \geq 0$ be l.s.c.. Show that

$$f = \sup\{g : g \in C_c(X),\ g \geq 0,\ g \leq f\}.$$

(Hint: Use Urysohn's lemma to construct, for $0 < a < f(x_0)$, $g(x_0) = a$, $g \in [0, a]$, etc.)

(3) Let X be a compact topological space. Show that every l.s.c function from X to $\mathbb{R}$ attains its minimum, that is, there exists some $x \in X$ such that $f(x) \leq f(y)$, $\forall y \in X$.

(4) Show that every semicontinuous function is a Borel function.

(5) Let $f : \mathbb{R}^n \to \mathbb{R}$ be Lebesgue measurable. Show that there exist Borel measurable functions g, h, $g(x) \leq f(x) \leq h(x)$ for all $x \in \mathbb{R}^n$ such that $g(x) = h(x)$ a.e.

(6) Let λ be a Borel measure and μ a Riesz measure on $\mathbb{R}^n$ such that $\lambda(G) = \mu(G)$ for all open sets G. Show that λ coincides with μ on $\mathcal{B}$.

(7) A characterization of the Lebsegue measure based on translational invariance. Let $(\mathbb{R}^n, \mathcal{B}, \mu)$ be a Borel measure space whose measure μ is translational invariant and is nontrivial in the sense that there exists some Borel set A such that $\mu(A) \in (0, \infty)$. Show that there exists a positive constant c such that $c\mu$ is the restriction of the Lebsegue measure on $\mathcal{B}$. Hint: First show that $\mu(C) = \mu(\overline{C})$ for every open cube C and then appeal to the problem above.

(8) Let K be compact in $\mathbb{R}^n$ and $K^\varepsilon = \{x : \text{dist}(x, K) < \varepsilon\}$ be open. Show that $\mathcal{L}^n(K^\varepsilon) \to \mathcal{L}^n(K)$ as $\varepsilon \to 0$.

(9) Let A and B be non-empty measurable sets in $\mathbb{R}^n$ such that $(1-\lambda)A + \lambda B$ is also measurable for all $\lambda \in (0, 1)$. Show that Brunn-Minkowski inequality is equivalent to either one of the following inequalities:

(a) $\mathcal{L}^n\big((1-\lambda)A + \lambda B\big) \geq (1-\lambda)\mathcal{L}^n(A) + \lambda\mathcal{L}^n(B)$.

(b) $\mathcal{L}^n\big((1-\lambda)A + \lambda B\big) \geq \min\big\{\mathcal{L}^n(A), \mathcal{L}^n(B)\big\}$.